JOURNAL OF APPROXIMATION THEORY 56, 316–329 (1989)

A Characteristic of Points in $\mathbb{R}^2$ Having Lebesgue Function of Polynomial Growth

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Communicated by E. W. Cheney

Received January 29, 1987

1. INTRODUCTION

Let $E \subset \mathbb{C}^m$ be compact. We wish to interpolate $f \in C(E)$ at points in E by a polynomial of degree n. There being $N_n := \binom{n+m}{m}$ monomials of degree at most n, this requires N_n interpolation points, $\mathbf{x}_i \in E$. We will often abbreviate $N_n = N$. Let $m_1(\mathbf{x}), m_2(\mathbf{x}), ..., m_N(\mathbf{x})$ be the appropriate monomials. We then desire coefficients, c_i, $1 \leqslant i \leqslant N$, such that $p_n(\mathbf{x}) := \sum_{i=1}^N c_i m_i(\mathbf{x})$ has the property that $p_n(\mathbf{x}_i) = f(\mathbf{x}_i)$, $1 \leqslant i \leqslant N$. The vector $\mathbf{c}$ may be expressed as the solution of a matrix equation as follows. Let M_n represent the matrix $[m_j(\mathbf{x}_i)] \in \mathbb{C}^{N \times N}$ and $\mathbf{f}$ the vector given by $f_i = f(\mathbf{x}_i)$. We must have

$$M_n \mathbf{c} = \mathbf{f}.$$

$V_n(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) := \det M_n$ is known as the Vandermonde determinant of the system. The interpolation problem has a unique solution if and only if $V_n(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) \neq 0$. An equivalent geometrical condition is that the points do not lie on an algebraic surface of degree n.

Now for a set of points for which $V \neq 0$ we may form the Lagrange polynomials, $l_i(\mathbf{x})$, defined by the conditions

$$l_i(\mathbf{x}_j) = \delta_{ij}, \qquad 1 \leqslant i, j \leqslant N.$$

The interpolating polynomial may then be expressed as

$$p_n(\mathbf{x}) = \sum_{i=1}^{N} f(\mathbf{x}_i)\, l_i(\mathbf{x}).$$

$\Lambda_n(\mathbf{x}) := \sum_{i=1}^N |l_i(\mathbf{x})|$ is known as the Lebesgue function of the inter-

0021-9045/89 $3.00

polation. It has the property that $\|p_n\|_E \leqslant \|\Lambda_n\|_E \|f\|_E$ and that if p_n^* is the best uniform approximation to $f \in C(E)$ on E, then

$$\|f - p_n\| \leqslant (1 + \|\Lambda_n\|_E) \|f - p_n^*\|. \tag{1.1}$$

For this reason it is desirable to find points for which $\|\Lambda_n\|_E$ is as small as possible. In one variable, for $E = [a, b]$, this problem is classical and the optmal interpolation points have been characterized by Kilgore [6] and deBoor and Pinkus [3]. From this characterization it follows that, for the optimal points, $\|\Lambda_n\| = O(\log n)$.

In several variables the characterization of the optimal points is evidently very difficult. A reasonable first step is to seek points for which the Lebesgue function grows polynomially in the sense that $\lim_{n\to\infty} \|\Lambda_n\|_E^{1/n} = 1$. Now, we may express

$$l_i(\mathbf{x}) = V_n(\mathbf{x}_1, ..., \mathbf{x}_{i-1}, \mathbf{x}, \mathbf{x}_{i+1}, ..., \mathbf{x}_N)/V_n(\mathbf{x}_1, ..., \mathbf{x}_N).$$

Hence, if the points $\mathbf{x}_1, ..., \mathbf{x}_N$ maximize $|V|$ as a function on E^N, we have, for $\mathbf{x} \in E$, $|l_i(\mathbf{x})| \leqslant 1$ and consequently $\|\Lambda_n\|_E \leqslant N$ and it is therefore of polynomial growth. This is almost surely a very pessimistic bound. In one variable, Sundermann [9] has shown that, in fact, the growth is logarithmic and experiments by Luttman and Rivlin [7] indicate that for the interval $[-1, 1]$, the Lebesgue function for the points which maximize the Vandermonde determinant is slightly smaller than that for the near optimal Chebyshev points.

Regardless, for any E this gives one example of points with the Lebesgue function of polynomial growth. In one variable many other examples are also known. For example, the roots of polynomials orthogonal with respect to a weighted L_2 inner product have this property. In contrast, little else seems to be known about the several variables case.

In this paper we restrict our attention to the case of $E = B_2 = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| \leqslant 1\}$, the unit disk in $\mathbb{R}^2$. The number of points and monomials is then $N = \binom{n+2}{2}$. We will show that for a set of points in B_2 having the Lebesgue function of polynomial growth,

$$\lim_{n\to\infty} V_n^{3/n(n+1)(n+2)} = (2e)^{-1/2}.$$

In one variable (see Goluzin [4]), the analogue of this limit yields what is known as the Chebyshev constant of E. For $E = [-1, 1]$ it is $\frac{1}{2}$, a reflection of the fact that the Chebyshev polynomials have norm $2^{-(n-1)}$. Of special interest is the fact that $(2e)^{-1/2} < \frac{1}{2}$.

2. Points for which the Lebesgue Function Grows Polynomially

If $E \subset \mathbb{C}$ is compact, then $\lim_{n\to\infty} \inf_{\mathbf{c}} \{ \|z^n - \sum_{k=0}^{n-1} c_k z^k\|_E \}^{1/n}$ exists and is known as the Chebyshev constant of E, $\tau(E)$. In the classical theory of transfinite diameter it is shown that if $z_1^{(n)}, z_2^{(n)}, ..., z_{n+1}^{(n)}$ are the points in E which maximize the Vandermonde determinant, then

$$\lim_{n\to\infty} |V(z_1^{(n)}, ..., z_{n+1}^{(n)})|^{1/\binom{n+2}{2}} = \tau(E).$$

The proofs of these facts and others may, for instance, be found in Goluzin [4]. In the case of $E = [-1, 1]$, it is known that $\tau(E) = \frac{1}{2}$.

More recently, Zaharajuta [10] has given a generalization of the above to the case of several variables. Suppose that the monomials are given explicitly, in multinomial notation, as $m_i(\mathbf{x}) := \mathbf{x}^{k(i)}$, and that the ordering is such that $i \leqslant j \Rightarrow |k(i)| \leqslant |k(j)|$; i.e., the ordering is consistent with the degree. Let $E \subset \mathbb{C}^m$ be compact and set

$$M_i := \inf_{\mathbf{c}} \left\{ \|p\|_E \colon p = m_i - \sum_{j=1}^{i-1} c_j m_j \right\}$$

and

$$\tau_i := M_i^{1/|k(i)|}.$$

If T is the standard $(m-1)$-simplex,

$$T_0 := \left\{ \boldsymbol{\theta} \in \mathbb{R}^m \colon \sum_{i=1}^{m} \theta_i = 1, \theta_i \geqslant 0, 1 \leqslant i \leqslant m \right\},$$

for $\boldsymbol{\theta} \in T$ let

$$\tau(\boldsymbol{\theta}) := \limsup_{\substack{j\to\infty \\ k(j)/|k(j)| \to \boldsymbol{\theta}}} \tau_j.$$

For $\boldsymbol{\theta}$ in the interior of T, Zaharjuta shows that the actual limit above exists. $\tau(\boldsymbol{\theta})$ is called the Chebyshev constant of E for the direction $\boldsymbol{\theta}$.

Now, in the case of one variable, the $\binom{n+2}{2}$ used in the exponent of the limit defining transfinite diameter is actually the degree of homogeneity of $V(x_1, ..., x_{n+1})$ considered as an $(n+1)$-variable polynomial. In several variables this degree of homogeneity may be computed to be

$$h_n := m \binom{n+m}{m+1}.$$

THEOREM 2.1 (Zaharjuta [10]). *Let $E \subset \mathbb{C}^m$ be compact. Suppose that $\mathbf{a}_1^{(n)}, \mathbf{a}_2^{(n)}, ..., \mathbf{a}_N^{(n)} \in E$ maximize $|V(\mathbf{x}_1, ..., \mathbf{x}_N)|$ over E^N. Then*

$$\lim_{n \to \infty} |V(\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)})|^{1/h_n} = \exp\left\{\frac{1}{\mathrm{vol}(T)} \int_T \log \tau(\boldsymbol{\theta})\, dV\right\}.$$

As stated, the theorem is for the points which maximize the Vandermonde determinant but because of the forgiving nature of the exponent, $1/h_n$, even more is true.

COROLLARY 2.2. *Let $E \subset \mathbb{C}^m$ be compact. Suppose that $\mathbf{x}_1^{(n)}, ..., \mathbf{x}_N^{(n)}$ is an array of points for which the Lebesgue function has polynomial growth. Then*

$$\lim_{n \to \infty} |V(\mathbf{x}_1^{(n)}, ..., \mathbf{x}_N^{(n)})|^{1/h_n} = \exp\left\{\frac{1}{\mathrm{vol}(T)} \int_T \log \tau(\boldsymbol{\theta})\, dV\right\}.$$

Proof. For simplicity write $\Lambda_n = \|\Lambda_n(x)\|_E$. Suppose that $\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)}$ are the points which maximize the Vandermonde determinant. We may regard $V(\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)})$ as a polynomial in the first variable, $\mathbf{a}_1^{(n)}$. Hence

$$V(\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)}) = \sum_{j=1}^{N} V(\mathbf{x}_j^{(n)}, \mathbf{a}_2^{(n)}, ..., \mathbf{a}_N^{(n)})\, l_j(\mathbf{a}_1^{(n)}),$$

where the l_j are the Lagrange polynomials associated with interpolation at the points $\mathbf{x}_i^{(n)}$. Therefore,

$$|V(\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)})| \leqslant \Lambda_n \max_{1 \leqslant j \leqslant N} |V(\mathbf{x}_j^{(n)}, \mathbf{a}_2^{(n)}, ..., \mathbf{a}_N^{(n)})|.$$

By relabelling if necessary we assume that

$$|V(\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)})| \leqslant \Lambda_n |V(\mathbf{x}_1^{(n)}, \mathbf{a}_2^{(n)}, ..., \mathbf{a}_N^{(n)})|.$$

We now consider $V(\mathbf{x}_1^{(n)}, \mathbf{a}_2^{(n)}, ..., \mathbf{a}_N^{(n)})$ as a polynomial in $\mathbf{a}_2^{(n)}$ and, after a possible further relabelling, obtain,

$$|V(\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)})| \leqslant \Lambda_n^2 |V(\mathbf{x}_1^{(n)}, \mathbf{x}_2^{(n)}, \mathbf{a}_3^{(n)}, ..., \mathbf{a}_N^{(n)})|.$$

Continuing in this manner we see that

$$|V(\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)})| \leqslant \Lambda_n^N |V(\mathbf{x}_1^{(n)}, \mathbf{x}_2^{(n)}, ..., \mathbf{x}_N^{(n)})|.$$

As the points $\boldsymbol{a}_i^{(n)}$ maximize $|V|$, we also have that

$$|V(\mathbf{x}_1^{(n)}, ..., \mathbf{x}_N^{(n)})| \leqslant |V(\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)})|$$

and therefore

$$\Lambda_n^{-N} |V(\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)})| \leqslant |V(\mathbf{x}_1^{(n)}, ..., \mathbf{x}_N^{(n)})| \leqslant |V(\mathbf{a}_1^{(n)}, ..., \mathbf{a}_N^{(n)})|.$$

But $N/h_n = \binom{n+m}{m} / \{m\binom{n+m}{m+1}\} = (m+1)/(mn)$ and thus, since by assumption $A_n^{1/n} \to 1$, $(A_n^{-N})^{1/h_n} \to 1$. The result follows. ▮

Our main result is to give an explicit value for the limit of the determinants in the case of the disk in $\mathbb{R}^2$. We will see that it is $(2e)^{-1/2}$. Note that this is less than the one variable value of $\frac{1}{2}$. Now let $B_m := \{\mathbf{x} \in \mathbb{R}^m : |\mathbf{x}| \leqslant 1\}$ be the unit ball in $\mathbb{R}^m$ and let S_{m-1} be its boundary. Any $f \in C(B_m)$ may also be regarded as a function on the sphere $S_m \subset \mathbb{R}^{m+1}$. Explicitly, evaluate $f(\mathbf{x}, x_{m+1}) = f(\mathbf{x})$. We set

$$\|f\|_2 := \left\{\frac{1}{\omega_m} \int_{S_m} f^2 \, d\sigma\right\}^{1/2}.$$

Here ω_m is the surface area of S_m. Our goal is to show that in the definition of $\tau(\boldsymbol{\theta})$ we may replace the uniform norm by this more tractable 2-norm. (Actually more general norms may be used but this is not needed here.)

We begin by showing that $\|\cdot\|_2$ and $\|\cdot\|_{B_m}$ are polynomially (in the degree) equivalent on the polynomials of degree at most n.

Lemma 2.3. *There are constants c_m, depending only on m, such that if p is a polynomial of degree at most n,*

$$\|p\|_2 \leqslant \|p\|_{B_m} \leqslant c_m n^{m/2} \|p\|_2.$$

Proof. The first inequality is straightforward. For the second, consider p as a function on S_m and express $p = \sum_{k=0}^n a_k Y_k$ where Y_k is the restriction to S_m of a homogeneous, harmonic polynomial of degree k (i.e., a spherical harmonic) that is normalized so that $\|Y_k\|_2 = 1$. It is known that $\int_{S_m} Y_k Y_j \, d\sigma = 0$ if $k \neq j$. Hence $\|p\|_2^2 = \sum_{k=0}^n a_k^2$. Now also

$$\|p\|_{B_m}^2 = \max_{\mathbf{x} \in S_m} \left| \sum_{k=0}^n a_k Y_k(\mathbf{x}) \right|^2 \leqslant \left(\sum_{k=0}^n a_k^2 \right) \max_{\mathbf{x} \in S_m} \sum_{k=0}^n Y_k^2(\mathbf{x}).$$

But from [8, Cor. 2.9, p. 144] it follows that

$$Y_k^2 \leqslant \frac{1}{\omega_m} \frac{m+2k-1}{k} \binom{m+k-2}{k-1} \leqslant d_m k^{m-1}$$

for some constant d_m. Hence

$$\|p\|_{B_m}^2 \leqslant c_m^2 n^m \|p\|_2^2$$

for some constant c_m and the result follows. ▮

Now, let

$$K_i := \inf_{\mathbf{c}} \left\{ \|p\|_2 : p = m_i - \sum_{j=1}^{i-1} c_j m_j \right\}$$

and

$$\mu_i := K_i^{1/|k(i)|}.$$

If p_i is the best uniform approximation to m_i on B_m and q_i is the best $\|\cdot\|_2$ approximation on B_m to m_i by polynomials of the form $\sum_{j=1}^{i-1} c_j m_j$, then

$$\|m_i - q_i\|_2 \leqslant \|m_i - p_i\|_2 \leqslant \|m_i - p_i\|_{B_m} = M_i$$

and

$$\|m_i - p_i\|_{B_m} \leqslant \|m_i - q_i\|_{B_m} \leqslant c_m n^{m/2} \|m_i - q_i\|_2.$$

Therefore,

$$\mu_i \leqslant \tau_i \leqslant (c_m)^{1/n} n^{m/(2n)} \mu_i. \tag{2.1}$$

We have used the abbreviation, $n = |k(i)|$. Note that as $n \to \infty$, $c_m^{1/n} n^{m/(2n)} \to 1$.

But, the error in best 2-norm approximation, K_i^2 in this case, may be expressed as the ratio of Gram determinants. From the above it is, therefore, not surprising that Zaharjuta's proof may be modified to yield:

THEOREM 2.4. *Let G_n be the Gram determinant of all monomials of degree at most n with respect to the inner product*

$$(f, g) := \frac{1}{\omega_m} \int_{S_m} fg \, d\sigma.$$

Then

$$\lim_{n \to \infty} G_n^{1/2h_n} = \exp \left\{ \frac{1}{\mathrm{vol}(T)} \int_T \log \tau(\boldsymbol{\theta}) \, dV \right\}.$$

Proof. Let $G^{(i)}$ denote the Gram determinant of monomials $m_1, m_2, \ldots, m_i$. Then

$$(\mu_i^{|k(i)|})^2 = K_i^2 = G^{(i)}/G^{(i-1)}.$$

Hence, by (2.1),

$$c_m^{-2} |k(i)|^{-m} (\tau_i^{|k(i)|})^2 \leqslant G^{(i)}/G^{(i-1)} \leqslant (\tau_i^{|k(i)|})^2,$$

and consequently,

$$(c_m^{-2}n^{-m})^{N_n-N_{n-1}}\left(\prod_{i=N_{n-1}+1}^{N_n}\tau_i^n\right)^2\leqslant G_n/G_{n-1}\leqslant\left(\prod_{i=N_{n-1}}^{N_n}\tau_i^n\right)^2.$$

Setting

$$\tau_n^{(0)}:=\left(\sum_{i=N_{n-1}+1}^{N_n}\tau_i\right)^{1/(N_n-N_{n-1})}$$

and

$$r_n:=n(N_n-N_{n-1}),$$

we have, as $G_0=1$,

$$(c_m^{-2}n^{-m})^{N_n-1}\left(\prod_{k=1}^{n}(\tau_k^{(0)})^{r_k}\right)^2\leqslant G_n\leqslant\left(\prod_{k=1}^{n}(\tau_k^{(0)})^{r_k}\right)^2.$$

Therefore,

$$(c_m^{-2}n^{-m})^{(N_n-1)/(2h_n)}\left(\prod_{k=1}^{n}(\tau_k^{(0)})^{r_k}\right)^{1/h_n}$$
$$\leqslant G_n^{1/(2h_n)}\leqslant\left(\prod_{k=1}^{n}(\tau_k^{(0)})^{r_k}\right)^{1/h_n}.$$

But, as before,

$$N_n/(2h_n)=(m+1)/(2mn)$$

and, hence, $(c_m^{-2}n^{-m})^{(N_n-1)/(2h_n)}\to 1$ as $n\to\infty$. We now rejoin Zaharjuta's proof in progress and we are done. ▮

We now compute G_n explicitly for the case of $m=2$.

THEOREM 2.5. *Let G_n be the Gram matrix of Theorem 2.4 with $m=2$. Then*

$$G_n=G_{n-1}\frac{1}{(2n+1)^{n+1}}\frac{2^{n(n+1)}}{\binom{2n}{n}^{2(n+1)}}\times\left\{\prod_{k=1}^{n/2}\binom{n+2k}{2k}\Big/\binom{n}{2k}\right\}^2,\qquad n \text{ even},$$

$$=G_{n-1}\frac{1}{(2n+1)^{n+1}}\frac{2^{n(n+1)}}{\binom{2n}{n}^{2(n+1)}}\times\left\{\prod_{k=0}^{(n-1)/2}\binom{n+2k+1}{2k+1}\Big/\binom{n}{2k+1}\right\}^2,\qquad n \text{ odd}.$$

Proof. The cases n even and n odd are only slighty different. We give the n even case only.

We make use of the spherical harmonics:

$$\begin{aligned}
&P_n(\cos(\theta)), \\
&\sin^m(\theta)\, P_n^{(m)}(\cos(\theta))\cos(m\varphi), \qquad 1 \leqslant m \leqslant n, \\
&\sin^m(\theta)\, P_n^{(m)}(\cos(\theta))\sin(m\varphi), \qquad 1 \leqslant m \leqslant n.
\end{aligned}$$

Here P_n is the nth Legendre polynomial and, in spherical coordinates, $z = \cos(\theta)$, $x = \sin(\theta)\cos(\varphi)$, and $y = \sin(\theta)\sin(\varphi)$. If $n - m$ is even, each of these is even in z and hence, by substituting $z^2 = 1 - x^2 - y^2$, we obtain $n+1$ bivariate polynomials of degree n which are orthogonal both to each other and all polynomials of lower degree with respect to the inner product of consideration.

Now if $\{q_1, q_2, ..., q_N\}$ is the set of polynomials so obtained, there is a matrix $T \in \mathbb{R}^{N \times N}$ giving the transition from the monomials $\{m_1, m_2, ..., m_N\}$, i.e., $\mathbf{q} = T\mathbf{m}$. Further, if Q_n is the (diagonal) Gram determinant of the q_i's then it is easy to see that

$$Q_n = |T|^2\, G_n, \qquad \text{or} \qquad G_n = Q_n/|T|^2. \tag{2.2}$$

But clearly T has the form

$$\begin{bmatrix} T_0 & \bigcirc & & \\ & T_1 & & \\ & & \ddots & \\ & * & & T_n \end{bmatrix}, \tag{2.3}$$

where $T_i \in \mathbb{R}^{(i+1)\times(i+1)}$ gives the degree i components of the degree i spherical harmonics.

As is well known,

$$\begin{aligned}
&\int_{S_2} P_n^2(\cos(\theta))\, d\sigma = 4\pi/(2n+1), \\
&\int_{S_2} \{\sin^m(\theta)\, P_n^{(m)}(\cos(\theta))\cos(m\varphi)\}^2\, d\sigma = \frac{2\pi}{(2n+1)}\frac{(n+m)!}{(n-m)!}, \\
&\int_{S_2} \{\sin^m(\theta)\, P_n^{(m)}(\cos(\theta))\cos(m\varphi)\}^2\, d\sigma = \frac{2\pi}{(2n+1)}\frac{(n+m)!}{(n-m)!}.
\end{aligned}$$

The degree n contribution to the diagonal determinant, Q_n, is the product of these over $1 \leqslant m \leqslant n$, $n-m$ even, i.e., as we assume n even, $m=2, 4, 6, ..., n$. This product is easily computed to be

$$\frac{1}{(2n+1)^{n+1}} \frac{1}{2^n} \left\{ \prod_{k=1}^{n/2} (n+2k)!/(n-2k)! \right\}^2. \tag{2.4}$$

Combining (2.2) with (2.3) and (2.4) we see that

$$G_n = G_{n-1} \times \frac{1}{(2n+1)^{n+1}} \frac{1}{2^n} \left\{ \prod_{k=1}^{n/2} (n+2k)!/(n-2k)! \right\}^2 \Big/ |T_n|^2 \tag{2.5}$$

and it remains to compute $|T_n|$.

Recall that $T_n \in \mathbb{R}^{(n+1)\times(n+1)}$ is the matrix of the coefficients of the exact degree n part of the degree n spherical harmonics with z^2 replaced by $1-x^2-y^2$. As the leading coefficient of P_n is $\binom{2n}{n}/2^n$, we have a contribution of

$$\binom{2n}{n}^{n+1} \Big/ 2^{n(n+1)}. \tag{2.6}$$

The derivatives $P_n^{(m)}$ have the additional factor, $n!/(n-m)!$, in the leading coefficient, giving an additional contribution to the determinant of

$$\left\{ \prod_{k=1}^{n/2} n!/(n-2k)! \right\}^2. \tag{2.7}$$

By Lemma 2.6, the determinant of what remains of the coefficients is

$$2^{n^2/2}. \tag{2.8}$$

Combining (2.6) with (2.7) and (2.8), we have that

$$|T_n| = 2^{-n(n+1)/2} \binom{2n}{n}^{n+1} \left\{ \prod_{k=1}^{n/2} n!/(n-2k)! \right\}^2.$$

We now substitute this expression for $|T_n|$ into (2.5) and simplify. ∎

LEMMA 2.6. *Let n be even. Let $A_n \in \mathbb{R}^{(n+1)\times(n+1)}$ be the matrix of the coefficients of the homogeneous polynomials*

$$(x^2+y^2)^{(n-m)/2} \sin^m(\theta) \cos(m\varphi), \qquad m=0, 2, 4, ..., n, \tag{2.9}$$

and

$$(x^2+y^2)^{(n-m)/2} \sin^m(\theta) \sin(m\varphi), \qquad m=2, 4, ..., n. \tag{2.10}$$

Here $x = \sin(\theta)\cos(\varphi)$, $y = \sin(\theta)\sin(\varphi)$, *and* $\sin^2(\theta) = x^2 + y^2$. *Then*

$$|\det A_n| = 2^{n^2/2}.$$

Proof. First notice that the polynomials of (2.9) and (2.10) consist of disjoint sets of monomials; i.e., (2.9) yields polynomials with the powers of x and y both even while (2.10) yields polynomials with both powers odd. Thus the determinant of A_n is the product of the determinants of B_n and C_n where $B_n \in \mathbb{R}^{(n/2+1)\times(n/2+1)}$ is the matrix of the coefficients of the polynomials (2.9) and $C_n \in \mathbb{R}^{(n/2)\times(n/2)}$ is the matrix of coefficients of the polynomials (2.10).

Consider first B_n. Letting $T_m(x) := \sum_{j=0}^{m/2} t_j x^{2j}$ denote the mth Chebyshev polynomial, we may write

$$\begin{aligned}
&(x^2+y^2)^{(n-m)/2}\sin^m(\theta)\cos(m\varphi)\\
&\quad= (x^2+y^2)^{(n-m)/2}\sin^m(\theta)\,T_m(\cos(\varphi))\\
&\quad= (x^2+y^2)^{(n-m)/2}\sum_{j=0}^{m/2} t_j\cos^{2j}(\varphi)\sin^{2j}(\theta)\sin^{m-2j}(\theta)\\
&\quad= (x^2+y^2)^{(n-m)/2}\sum_{j=0}^{m/2} t_j x^{2j}(x^2+y^2)^{(m-2j)/2}\\
&\quad= \sum_{j=0}^{m/2} t_j x^{2j}(x^2+y^2)^{(n-2j)/2}\\
&\quad= (x^2+y^2)\sum_{j=0}^{m/2} t_j x^{2j}(x^2+y^2)^{(n-2-2j)/2}. \qquad (2.11)
\end{aligned}$$

We see that each polynomial is obtained from one of degree $(n-2)$ by multiplying by (x^2+y^2). The exception is the case $m=n$ which does not occur for lower degrees. Perhaps this is best illustrated by an example.

For $n=4$ the polynomials are

$$\begin{aligned}
m=0:&\quad x^4 + 2x^2y^2 + y^4\\
m=2:&\quad x^4 + - y^4 \qquad (2.12)\\
m=4:&\quad x^4 - 6x^2y^2 + y^4
\end{aligned}$$

and for $n=6$ they are

$$\begin{aligned}
m=0:&\quad x^6 + 3x^4y^2 + 3x^2y^4 + y^6\\
m=2:&\quad x^6 + x^4y^2 - x^2y^4 - y^6\\
m=4:&\quad x^6 - 5x^4y^2 - 5x^2y^4 + y^6\\
m=6:&\quad x^6 - 15x^4y^2 + 15x^2y^4 - y^6.
\end{aligned} \qquad (2.13)$$

Observe that the first three of (2.13) are those of (2.12) multiplied by (x^2+y^2). The last of (2.13) is new.

If we write the monomials in descending powers of x, we have the matrices

$$B_4=\begin{bmatrix}1 & 2 & 1\\ 1 & 0 & -1\\ 1 & -6 & 1\end{bmatrix} \text{ and } \quad B_6=\begin{bmatrix}1 & 3 & 3 & 1\\ 1 & 1 & -1 & -1\\ 1 & -5 & -5 & 1\\ 1 & -15 & 15 & -1\end{bmatrix}.$$

Because of (2.11), *except for the last entry* in columns 2 through $n/2$ of B_n, each such interior column is the sum of the same column and the one preceding it of B_{n-2}. Further, in each polynomial, the coefficient of x^n is $\sum_{j=0}^{n/2} t_j = T_m(1)=\cos(\cos^{-1}(1))=1$ and the coefficient of y^n is $t_0=T_m(0)=(-1)^{m/2}$. Hence the first and last columns of the matrices B_n, for various n, are simply extensions of each other.

By the above remarks we see that upon subtracting col 2 − col 1, col 3 − col 2, col 4 − col 3, ..., col$(n/2+1)$ − col$(n/2)$, in this order, we have

$$\det B_n = \det\begin{bmatrix} B_{n-2} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ * & r\end{bmatrix},$$

where $r=\sum_{k=1}^{(n+2)/2}(-1)^{k-1}b_{n/2+1,k}$, i.e., the alternating sum of the bottom row of B_n. But the bottom row corresponds to the case $m=n$ and therefore consists of the coefficients of the polynomial

$$p(x,y):=\sum_{j=0}^{n/2} t_j x^{2j}(x^2+y^2)^{(n-2j)/2}$$

and r is the alternating sum of the coefficients of $p(x,y)$.

As p is homogeneous of degree $n/2$,

$$\begin{aligned} r=p(i,1)&=\sum_{j=0}^{n/2} t_j(-1)^j(-1+1)^{(n-2j)/2}\\ &=t_{n/2}=2^{n-1}.\end{aligned}$$

Hence $|\det B_n|=2^{n-1}|\det B_{n-2}|$ and since an easy calculation reveals that $|\det B_2|=2$, we see that $|\det B_n|=2^{n^2/4}$.

An argument exactly analogous to the above shows also that $|\det C_n|=2^{n^2/4}$ and the result follows. ∎

This explicit expression allows us to compute a numerical value for $\lim_{n\to\infty} G_n^{1/h_n}$. Note that in two dimensions $h_n = n(n+1)(n+2)/3$.

THEOREM 2.6. *Let G_n be the Gram determinant of Theorem 2.4 with $m=2$. Then*

$$\lim_{n\to\infty} G_n^{1/h_n} = 1/(2e).$$

Proof. From Theorem 2.4 we know that the limit exists. There are two cases: n even and n odd. Their analyses being similar we give the proof for n even only.

Consider first the factor $\{\prod_{k=1}^{n/2} \binom{n+2k}{2k}\}^2$. Since we raise G_n to the extremely forgiving power $1/h_n$, it suffices to consider $\prod_{k=1}^{n} \binom{n+k}{k}$. Now

$$\log \prod_{k=1}^{n} \binom{n+k}{k} = \sum_{k=1}^{n} \log \frac{(n+k)!}{n!\,k!}$$

$$= \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n+k} \log(j) - \sum_{j=1}^{n} \log(j) - \sum_{j=1}^{k} \log(j) \right\},$$

which after some manipulation reduces to

$$(2n+1) \sum_{j=1}^{2n} \log(j) - (3n+2) \sum_{j=1}^{n} \log(j)$$
$$- \sum_{j=1}^{2n} j\log(j) + 2 \sum_{j=1}^{n} j\log(j). \qquad (2.14)$$

But by Euler's summation formula,

$$\sum_{j=1}^{n} \log(j) = n\log(n) - n + \frac{1}{2}\log(n) + O(1) \qquad (2.15)$$

and

$$\sum_{j=1}^{n} j\log(j) = \frac{1}{2} n^2 \log(n) + \frac{-1}{4} n^2 + \frac{1}{2} n\log(n) + \frac{1}{12}\log(n) + O(1). \qquad (2.16)$$

It follows from (2.14) that

$$\log \prod_{k=1}^{n} \binom{n+k}{k} = \left(2\log(2) - \frac{1}{2}\right) n^2 + O(n\log(n)). \qquad (2.17)$$

Second, consider $\{\prod_{k=1}^{n/2} \binom{n}{2k}\}^2$. Again, it suffices to study $\prod_{k=1}^{n} \binom{n}{k}$. But

$$\log \prod_{k=1}^{n} \binom{n}{k} = \sum_{k=1}^{n} \log \frac{n!}{(n-k)!\,k!}$$
$$= \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} \log(j) - \sum_{j=1}^{n-k} \log(j) - \sum_{j=1}^{k} \log(j) \right\},$$

which after some manipulation reduces to

$$-(n+1) \sum_{j=1}^{n} \log(j) + 2 \sum_{j=1}^{n} j \log(j).$$

Using (2.15) and (2.16) we thus see that

$$\log \prod_{k=1}^{n} \binom{n}{k} = \frac{1}{2} n^2 + O(n \log(n)). \tag{2.18}$$

A similar calculation reveals that

$$\log \binom{2n}{n}^{2(n+1)} = (4 \log(2))\, n(n+1) + O(n \log(n)). \tag{2.19}$$

Combining (2.17), (2.18), and (2.19) we see that

$$\log \left\{ \frac{1}{(2n+1)^{n+1}} \frac{2^{n(n+1)}}{\binom{2n}{n}^{2(n+1)}} \left\{ \prod_{k=1}^{n/2} \binom{n+2k}{2k} \Big/ \binom{n}{2k} \right\}^2 \right\}$$
$$= -(1 + \log(2))\, n^2 + O(n \log(n)). \tag{2.20}$$

But $3/n(n+1)(n+2) \sum_{k=1}^{n} k \log(k) \to 0$ as $n \to \infty$. Therefore

$$\lim_{n \to \infty} \frac{3}{n(n+1)(n+2)} \log G_n = \lim_{n \to \infty} \frac{-3(1+\log 2)}{n(n+1)(n+2)} \sum_{k=1}^{n} k^2$$
$$= -(1 + \log(2)).$$

The taking of exponentials gives the result. ∎

We may summarize our results as follows.

THEOREM 2.7. *Suppose that* $\mathbf{x}_1^{(n)}, ..., \mathbf{x}_N^{(n)} \in B_2 \subset \mathbb{R}^2$ *form an array of points for which the Lebesgue function has polynomial growth. Then*

$$\lim_{n \to \infty} |V(\mathbf{x}_1^{(n)}, ..., \mathbf{x}_N^{(n)})|^{1/h_n} = 1/\sqrt{2e}.$$

This gives a specific numerical characteristic of points in the disk for which the Lebesgue function has polynomial growth.

REFERENCES

1. M. ABRAMOWITZ AND I. A. STEGUN, "Handbook of Mathematical Function," Dover, New York.
2. L. BOS, "Near Optimal Location of Points for Lagrange Interpolation in Several Variables," Ph.D. thesis, University of Toronto, 1981.
3. C. DEBOOR AND A. PINKUS, Proof of the conjecture of Bernstein and Erdös concerning the optimal nodes for polynomial interpolation, *J. Approx. Theory* **24** (1978), 289–303.
4. G. GOLUZIN, "Geometric Theory of Functions of a Complex Variable," *Amer. Math. Soc.*, Providence, RI, 1969.
5. I. S. GRADSHTEYN AND I. M. RYZHIK, "Tables of Integrals, Series, and Products," Academic Press, New York, 1965.
6. T. A. KILGORE, A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm, *J. Approx. Theory* **24** (1978), 273–288.
7. F. W. LUTTMAN AND T. J. RIVLIN, Some numerical experiments in the theory of polynomial interpolation, *IBM J. Res. Develop.* **9** (1965), 187–191.
8. E. STEIN AND G. WEISS, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, NJ, 1971.
9. B. SUNDERMANN, Lebesgue Constants in Lagrangian Interpolation at the Fekete Points, *Ergeb. Lehrstuhle Math III und VIII*, Nr. **44** (1980), Univ. Dortmund.
10. V. ZAHARJUTA, Transfinite Diameter, Chebyshev Constants and Capacity for Compacta in $\mathbb{C}^n$, *Math. USSR-Sb.* **25**, No. 3, (1975), 350–364.